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Dimension Spectra of Hyperbolic Flows

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Abstract For flows with a conformal hyperbolic set, we establish a conditional variational principle for the dimension spectra of Hölder continuous functions. We consider *simultaneously* Birkhoff averages into the future and into the past. We emphasize that the description of the spectra is not a consequence of the existing results for Birkhoff averages into the future (or into the past). The main difficulty is that even though the local product structure is bi-Lipschitz, the level sets of the Birkhoff averages are never compact. Our proof is based on the use of Markov systems and is inspired in earlier arguments in the case of discrete time.

Keywords Dimension spectrum · Hyperbolic flow

1 Introduction

1.1 Multifractal Analysis and Dimension Spectra

The theory of multifractal analysis can be considered a subfield of the dimension theory of dynamical systems. Essentially, it studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. In particular, we can consider Birkhoff averages, Lyapunov exponents, pointwise dimensions, and local entropies. We emphasize that these functions are usually only measurable and thus their level sets are rarely manifolds.

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Hence, in order to measure their complexity it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension. The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia and Shraiman in [18]. The first rigorous approach is due to Collet, Lebowitz and Porzio in [12] for a class of measures invariant under one-dimensional Markov maps. In [30], Lopes considered the measure of maximal entropy for hyperbolic Julia sets, and in [39], Rand studied Gibbs measures for a class of repellers. We refer the reader to the books [1, 35] for further references and detailed expositions of parts of the theory.

We briefly recall the main components of multifractal analysis. Let $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ be a flow in M preserving a finite measure μ. By Birkhoff's ergodic theorem, for each μ-integrable function $a\colon M \to \mathbb{R}$ the limit

$$a_\Phi(x) = \lim_{t\to\infty} \frac{1}{t} \int_0^t a(\varphi_s(x))\, ds$$

exists for μ-almost every $x \in M$. For each $\alpha \in \mathbb{R}$ we consider the level set

$$K_\alpha^+(a) = \{x \in M : a_\Phi(x) = \alpha\},$$

i.e., the set of points $x \in M$ such that $a_\Phi(x)$ is well-defined and is equal to α. We also consider the set

$$K(a) = \left\{x \in M : \liminf_{t\to\infty} \frac{1}{t} \int_0^t a(\varphi_s(x))\, ds < \limsup_{t\to\infty} \frac{1}{t} \int_0^t a(\varphi_s(x))\, ds\right\}.$$

Clearly,

$$M = K(a) \cup \bigcup_{\alpha\in\mathbb{R}} K_\alpha^+(a). \tag{1}$$

We call the decomposition of M in (1) a *multifractal decomposition*. One way to measure the complexity of the sets $K_\alpha^+(a)$ is to compute their Hausdorff dimension. Namely, the *dimension spectrum*

$$D\colon \{\alpha \in \mathbb{R} : K_\alpha^+(a) \neq \emptyset\} \to \mathbb{R}$$

is defined by

$$D(\alpha) = \dim_H K_\alpha^+(a),$$

where $\dim_H A$ denotes the Hausdorff dimension of the set A. The dimension spectra of conformal hyperbolic sets of a flow were described by Pesin and Sadovskaya in [36]. One can also consider other characteristics to measure the complexity of the level sets. For example, we obtain the entropy spectra by considering the topological entropy of Φ in $K_\alpha^+(a)$. The entropy spectra of hyperbolic sets of a flow were described by Barreira and Saussol in [7].

Our main objective is to give a complete description of the dimension spectra of Birkhoff averages in a conformal locally maximal hyperbolic set of a flow, taking *simultaneously* into account Birkhoff averages into the future and into the past. More precisely, the spectra that we consider are obtained by computing the Hausdorff dimension of the level sets of Birkhoff averages of a given function both for positive and negative time. Namely, we also consider the level sets

$$K_\beta^-(b) = \left\{x \in M : \lim_{t\to-\infty} \frac{1}{t} \int_0^t b(\varphi_s(x))\, ds = \beta\right\}.$$

Our main aim is thus to describe the multifractal spectrum

$$(\alpha, \beta) \mapsto \dim_H(K_\alpha^+(a) \cap K_\beta^-(b)),$$

and in particular to show that it is analytic in the interior of its domain (see Theorem 1). For flows with a conformal locally maximal hyperbolic set, using the fact that the stable (respectively unstable) local manifold of a given point has exactly the same future Birkhoff average (respectively past Birkhoff average) of that point, we show that the level sets $K_\alpha^+(a) \cap K_\beta^-(b)$ have a local product structure (in the same manner as the hyperbolic set does). This is the main observation that allows us to use a conditional variational principle in [2] to describe the dimension spectrum. On the other hand, the main difficulty is that the level sets are not compact. This leads to the construction of *noninvariant* measures concentrated on each level set and with the appropriate pointwise dimension. We also consider the higher-dimensional case of more than one Birkhoff average, as well as the case of ratios of Birkhoff averages.

We note that Theorem 1 was formulated earlier in [2] but there is a gap in the proof (more precisely in the proof of Theorem 14). We now present an alternative proof based on arguments of Barreira and Valls in [9], using also results of Barreira and Saussol in [7]. This fixes the gap in [2].

1.2 Multifractal Analysis and Statistical Mechanics

We discuss in this section a few selected topics illustrating the relevance of multifractal analysis for statistical mechanics, while also referring to a few recent works.

We first mention that one can consider several statistical models defined in terms of Farey fractions. For example, following [13], among other models one can consider the so-called Farey fraction spin chain. This is a one-dimensional statistical model proposed by Kleban and Özlük in [29], which can be described as a periodic chain of sites with two possible spin states at each site. We note that the model allows phase transitions. On the other hand, it is well know that Farey fractions appear in connection with the study of multifractals associated to some chaotic maps exhibiting intermittency. From the mathematical point of view, this is also related to the study of nonuniformly hyperbolic systems and countable topological Markov chains. It turns out that for uniformly hyperbolic systems and their codings by finite topological Markov chains the dimension and entropy spectra of an equilibrium measure of a Hölder continuous function has bounded domain and is analytic. In strong contrast, in the case of nonuniformly hyperbolic systems and countable topological Markov chains the spectrum may have unbounded domain and need not be analytic. In [38], Pollicott and Weiss presented a multifractal analysis of the Lyapunov exponent for the Gauss map and for the Manneville–Pomeau transformation. Related results were obtained by Yuri in [48]. In [31–33], Mauldin and Urbański developed the theory of infinite conformal iterated function systems, studying in particular the Hausdorff dimension of the limit set (see also [19]). Related results were obtained by Nakaishi in [34]. In [28], Kesseböhmer and Stratmann established a detailed multifractal analysis for Stern–Brocot intervals, continued fractions, and certain Diophantine growth rates, building on their former work [27]. In particular, they discussed multifractal spectra closely related to the Farey map and to the Gauss map. We refer to [37] for other results concerning Farey trees and multifractal analysis. In [21], Iommi obtained a detailed multifractal analysis for countable topological Markov chains, using the so-called Gurevich pressure introduced by Sarig in [43] (building on former work of Gurevich [17] on the notion of topological entropy for countable Markov chains). In [3], Barreira and Iommi considered the case of suspension flows over a countable topological

Markov chain, building also on work of Savchenko [44] on the notion of topological entropy. In [22], Iommi and Skorulski studied the multifractal analysis of conformal measures for the exponential family $z \mapsto \lambda e^z$ with $\lambda \in (0, 1/e)$ (we note that in this setting the Julia set J is not compact and that the dynamics is not Markov on J). They use a construction described by Urbański and Zdunik in [45].

Another direction of research is related to the so-called multifractal rigidity, which uses the notion of topological pressure introduced by Ruelle in [41] for expansive maps, and by Walters in [46] in the general case. Namely, it is believed by some specialists (see for example [35]) that the information encoded by the multifractal spectra can be used to recover the potential and somehow also the dynamics (possibly up to some appropriate equivalence). This is essentially due to a relation that often occurs between a multifractal spectrum and a certain function obtained from the potential using the topological pressure (see Sect. 2 for the definitions). Namely, in some situations these two functions form a Legendre pair, and this allows one to try to obtain information about the potential (perhaps up to some equivalence) from the information encoded in the multifractal spectrum (see [1, 35] for introductions to the theory of multifractal analysis, and for the description of some recent results). This approach is particularly welcome when the multifractal spectra can be determined with arbitrary precision, while this may not be the case with the dynamical system, which may not be known a priori or at least may not be known with arbitrary precision. We remark that instead of dealing with local quantities associated to a given trajectory, we deal here with quantities of global nature, which are encoded in the multifractal spectra. The phenomenon of multifractal rigidity occurs when for two topologically equivalent dynamical systems with identical multifractal spectra, the original potentials are equivalent (in some sense that needs to be made precise in each case, for example up to some conjugacy of the dynamics). This leads to a "multifractal classification" of the dynamics (either invertible or noninvertible) in terms of the multifractal spectra, and we may be able to recover information about a potential from the information encoded in its multifractal spectra. Related results were obtained by Barreira, Pesin and Schmeling in [4, 5] for some classes of uniformly hyperbolic dynamical systems. We note that, in general, when we use a single spectrum there is no multifractal rigidity even for topological Markov chains with 3 symbols (see [6]). However, this does not forbid the occurrence of the multifractal rigidity phenomenon for other classes of potentials and other classes of dynamics. We refer to [1] for a related detailed discussion.

In another direction, in [16] Frisch and Matsumoto established the multifractality of the Feigenbaum invariant measure appearing at the accumulation point of the period doubling cascade. They also used a thermodynamic formalism. More recently, in [15] Frisch, Khanin and Matsumoto also considered this measure and they provided numerical evidence that some related fractional derivatives have power-law tails in their cumulative distributions, whose exponents are related to what they call the spectrum of singularities. A related mathematical theory that bears some resemblance is due to Jaffard in [23, 24] who developed a multifractal analysis for functions, essentially looking at the best Hölder exponent of a function at each given point. In this respect, we also want to mention the two papers [25, 26] by Jaffard and Mélot developing tools to study the local behavior of the boundary of a domain in a finite-dimensional space. In particular, they study the boundary in terms of the dimension of certain level subsets. Again their approach bears a resemblance to the more standard multifractal analysis involving the thermodynamic formalism.

Still in another direction, Fisch studied in [14] the ground state entropy of the 2D Ising spin glass with $+1$ and -1 bonds for $L \times M$ square lattices and with fraction of negative bonds equal to 0.5, using periodic and/or antiperiodic boundary conditions. He obtained the domain wall entropy as a function of L and M. For the zero-energy domain walls, he argued that the probability distribution of the domain wall entropy is multifractal, as M/L^d becomes large with $d = 1.22 \pm 0.01$, as a result of disorder-induced localization.

2 Basic Notions of the Thermodynamic Formalism

2.1 Topological Pressure and Entropy

We recall several basic notions of the thermodynamic formalism for flows. We refer to [7, 42, 47] for details.

Let $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow in a compact metric space (X, d). Given $x \in X$, $t > 0$, and $\varepsilon > 0$, we define

$$B_\varepsilon(x,t) = \big\{y \in X : d(\varphi_s(y), \varphi_s(x)) < \varepsilon \text{ for every } s \in [0,t]\big\}.$$

Now let $a\colon X \to \mathbb{R}$ be a continuous function. We write

$$a(x,t,\varepsilon) = \sup\left\{\int_0^t a(\varphi_s(y))\,ds : y \in B_\varepsilon(x,t)\right\}. \tag{2}$$

Given a set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M(Z,a,\alpha,\varepsilon) = \lim_{T\to\infty} \inf_\Gamma \sum_{(x,t)\in\Gamma} \exp\big[a(x,t,\varepsilon) - \alpha t\big],$$

where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i, t_i)\}_{i\in I}$ such that $x_i \in X$ and $t_i \ge T$ for every $i \in I$, with $\bigcup_{i\in I} B_\varepsilon(x_i, t_i) \supset Z$. One can show that the limit

$$P_\Phi(a|Z) := \lim_{\varepsilon\to 0} \inf\big\{\alpha \in \mathbb{R} : M(Z,a,\alpha,\varepsilon) = 0\big\}$$

exists. The number $P_\Phi(a|Z)$ is called the *topological pressure of a on the set Z* (with respect to the flow Φ). We also write $P_\Phi(a) = P_\Phi(a|X)$. The number $h(\Phi|Z) = P_\Phi(0|Z)$ is called the *topological entropy of Φ on Z*.

Now we consider the set $\mathcal{M}_\Phi(X)$ of all Φ-invariant probability measures in X. We recall that a measure μ in X is said to be *Φ-invariant* if $\mu(\varphi_t(A)) = \mu(A)$ for every $t \in \mathbb{R}$ and every measurable set $A \subset X$. With the weak$*$ topology the space $\mathcal{M}_\Phi(X)$ is compact and metrizable. A measure μ in X is said to be *ergodic* if for every Φ-invariant set $A \subset X$ (i.e., such that $\varphi_t(A) = A$ for every $t \in \mathbb{R}$) we have $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. We define

$$h(Z,\varepsilon) = \inf\big\{\alpha \in \mathbb{R} : M(Z,0,\alpha,\varepsilon) = 0\big\}. \tag{3}$$

One can show that for each measure $\mu \in \mathcal{M}_\Phi(X)$ the limit

$$h_\mu(\Phi) := \lim_{\varepsilon\to 0} \inf\big\{h(Z,\varepsilon) : \mu(Z) = 1\big\} \tag{4}$$

exists. Moreover, we have the following result.

Proposition 1 *If Φ is a continuous flow in a compact metric space X, and $\mu \in \mathcal{M}_\Phi(X)$ is ergodic, then $h_\mu(\Phi)$ coincides with the Kolmogorov–Sinai entropy of Φ with respect to μ.*

Therefore, in the case of ergodic measures we can use (3)–(4) to compute the entropy. An analogous statement in the case of discrete time was established by Pesin in [35, Theorem 11.6]. The proof of Proposition 1 is a simple modification of the proof of that statement and hence it is not given here.

We also recall the variational principle for the topological pressure.

Proposition 2 *If Φ is a continuous flow in a compact metric space X, and $a\colon X \to \mathbb{R}$ is a continuous function, then*

$$P_\Phi(a) = \sup\left\{h_\mu(\Phi) + \int_X a\,d\mu : \mu \in \mathcal{M}_\Phi(X)\right\}. \tag{5}$$

A measure $\mu \in \mathcal{M}_\Phi(X)$ is called an *equilibrium measure* for the function a (with respect to the flow Φ) if the supremum in (5) is attained at this measure, i.e., if

$$P_\Phi(a) = h_\mu(\Phi) + \int_X a\,d\mu.$$

2.2 u-dimension

Now we recall a Carathéodory dimensional characteristic introduced by Barreira and Saussol in [7]. This notion is a generalization of the topological entropy, and is also a version of the Carathéodory dimensional characteristic introduced by Barreira and Schmeling in [8] in the case of discrete time.

Let Φ be a continuous flow in a compact metric space X, and let $u\colon X \to \mathbb{R}$ be a positive continuous function. Given a set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$N(Z, u, \alpha, \varepsilon) = \lim_{T\to\infty} \inf_\Gamma \sum_{(x,t)\in\Gamma} \exp(-\alpha u(x, t, \varepsilon)),$$

with $u(x, t, \varepsilon)$ as in (2), where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i, t_i)\}_{i\in I}$ such that $x_i \in X$ and $t_i \geq T$ for every $i \in I$, with $\bigcup_{i\in I} B_\varepsilon(x_i, t_i) \supset Z$. We also define

$$\dim_{u,\varepsilon} Z = \inf\{\alpha \in \mathbb{R} : N(Z, u, \alpha, \varepsilon) = 0\}.$$

One can show that the limit

$$\dim_u Z := \lim_{\varepsilon\to 0} \dim_{u,\varepsilon} Z$$

exists. The number $\dim_u Z$ is called the *u-dimension of the set Z* (*with respect to u*). Clearly, when $u = 1$ we have $\dim_u Z = h(\Phi|Z)$.

It follows easily from the definitions that the topological pressure and the u-dimension are related as follows.

Proposition 3 *We have $P_\Phi(-\alpha u|Z) = 0$ if and only if $\alpha = \dim_u Z$.*

For each probability measure μ in X, we define

$$\dim_{u,\varepsilon} \mu = \inf\{\dim_{u,\varepsilon} Z\colon \mu(Z) = 1\}.$$

One can show that the limit

$$\dim_u \mu := \lim_{\varepsilon \to 0} \dim_{u,\varepsilon} \mu$$

exists. The number $\dim_u \mu$ is called the *u-dimension of μ (with respect to u)*. For each ergodic measure $\mu \in \mathcal{M}_\Phi(X)$ we have

$$\dim_u \mu = h_\mu(\Phi) \Big/ \int_X u \, d\mu.$$

The proof of this identity can be obtained in an analogous manner to the one in the case of discrete time in [8].

3 Hyperbolic Flows and Multifractal Spectra

3.1 Basic Notions

Given a C^1 flow $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ in a smooth manifold M, a compact Φ-invariant set $\Lambda \subset M$ is said to be *hyperbolic* (for the flow Φ) if there are a decomposition

$$T_\Lambda M = E^s \oplus E^u \oplus E^0,$$

and constants $c > 0$ and $\lambda \in (0, 1)$ such that for every $x \in \Lambda$:

1. the vector $\frac{d}{dt}(\varphi_t(x))|_{t=0}$ generates $E^0(x)$;
2. for every $t \in \mathbb{R}$ we have

$$d_x\varphi_t E^s(x) = E^s(\varphi_t(x)) \quad \text{and} \quad d_x\varphi_t E^u(x) = E^u(\varphi_t(x));$$

3. $\|d_x\varphi_t v\| \le c\lambda^t \|v\|$ for every $v \in E^s(x)$ and $t > 0$;
4. $\|d_x\varphi_{-t} v\| \le c\lambda^t \|v\|$ for every $v \in E^u(x)$ and $t > 0$.

For example, for any geodesic flow in a compact Riemannian manifold with negative sectional curvature the unit tangent bundle is a hyperbolic set. Moreover, time-changes and small C^1 perturbations of flows with a hyperbolic set have also a hyperbolic set.

We say that a hyperbolic set Λ (for the flow Φ) is *locally maximal* if it has an open neighborhood U such that $\Lambda = \bigcap_{t \in \mathbb{R}} \varphi_t(U)$. The flow Φ is said to be *topologically mixing on* Λ if for any nonempty open sets U and V intersecting Λ, there exists $t \in \mathbb{R}$ such that $\varphi_s(U) \cap V \cap \Lambda \ne \emptyset$ for every $s > t$. Finally, a function $a \colon \Lambda \to \mathbb{R}$ is said to be *Φ-cohomologous* to a function $b \colon \Lambda \to \mathbb{R}$ if there is a bounded measurable function $q \colon \Lambda \to \mathbb{R}$ such that

$$a(x) - b(x) = \lim_{t \to 0} \frac{q(\varphi_t(x)) - q(x)}{t} \tag{6}$$

for every $x \in \Lambda$, in which case $P_\Phi(a) = P_\Phi(b)$.

We have the following properties (see [42, 47]).

Proposition 4 *For a C^1 flow Φ, let Λ be a locally maximal hyperbolic set such that Φ is topologically mixing on Λ. Then*:

1. *the function $\mu \mapsto h_\mu(\Phi)$ is upper semi-continuous in $\mathcal{M}_\Phi(\Lambda)$*;
2. *each Hölder continuous function $a \colon \Lambda \to \mathbb{R}$ has a unique equilibrium measure*;

3. *given Hölder continuous functions* $a, b\colon \Lambda \to \mathbb{R}$, *the function* $\mathbb{R} \ni t \mapsto P_\Phi(a+tb)$ *is analytic, and for each* $t \in \mathbb{R}$ *we have*

$$\frac{d^2}{dt^2} P_\Phi(a+tb) \geq 0,$$

with equality if and only if b *is* Φ*-cohomologous to a constant.*

3.2 Multifractal Spectra

We denote by $C^\delta(\Lambda)$ the space of Hölder continuous functions in Λ with Hölder exponent $\delta \in (0,1]$. Given $d \in \mathbb{N}$ we set $F(\Lambda) = C^\delta(\Lambda)^d \times C^\delta(\Lambda)^d$. Now we consider vectors

$$A = (a_1, \ldots, a_d) \quad \text{and} \quad B = (b_1, \ldots, b_d),$$

in $F(\Lambda)$, with $b_i > 0$ for $i = 1, \ldots, d$ (for simplicity we simply write $B > 0$). Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ we set

$$K_\alpha = \bigcap_{i=1}^{d} \left\{ x \in \Lambda : \lim_{t\to\infty} \frac{\int_0^t a_i(\varphi_s(x))ds}{\int_0^t b_i(\varphi_s(x))ds} = \alpha_i \right\}.$$

Now let $u\colon \Lambda \to \mathbb{R}$ be a positive continuous function. We define the *u-dimension spectrum* $\mathcal{F}_u\colon \mathbb{R}^d \to \mathbb{R}$ *of the pair* (A, B) (with respect to Φ) by

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha.$$

We also consider the function $\mathcal{P}\colon \mathcal{M}_\Phi(\Lambda) \to \mathbb{R}^d$ defined by

$$\mathcal{P}(\mu) = \left(\frac{\int_\Lambda a_1\, d\mu}{\int_\Lambda b_1\, d\mu}, \ldots, \frac{\int_\Lambda a_d\, d\mu}{\int_\Lambda b_d\, d\mu} \right).$$

Finally, given $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ in $\mathbb{R}^d$ we write

$$\alpha * \beta = (\alpha_1\beta_1, \ldots, \alpha_d\beta_d) \in \mathbb{R}^d \quad \text{and} \quad \langle \alpha, \beta \rangle = \sum_{i=1}^{d} \alpha_i \beta_i \in \mathbb{R}.$$

The following result was established in [2]. In particular, it provides a conditional variational principle for the spectrum $\mathcal{F}_u$.

Proposition 5 (See [2, Theorems 6 and 10]) *Let* Φ *be a* C^1 *flow with a locally maximal hyperbolic set* Λ *on which* Φ *is topologically mixing, and let* $(A, B) \in F(\Lambda)$ *with* $B > 0$ *and* $u \in C^\delta(\Lambda)$ *with* $u > 0$. *If* $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M}_\Phi(\Lambda))$, *then* $K_\alpha \neq \emptyset$ *and the following properties hold*:

1. $\mathcal{F}_u(\alpha)$ *satisfies the conditional variational principle*

$$\mathcal{F}_u(\alpha) = \max \left\{ \frac{h_\mu(\Phi)}{\int_\Lambda u\, d\mu} : \mu \in \mathcal{M}_\Phi(\Lambda) \text{ and } \mathcal{P}(\mu) = \alpha \right\};$$

2. $\mathcal{F}_u(\alpha) = \min\{T_u(\alpha, q) : q \in \mathbb{R}^d\}$, *where* $T_u(\alpha, q)$ *is the unique real number such that*

$$P_\Phi(\langle q, A - \alpha * B \rangle - T_u(\alpha, q)u) = 0;$$

3. *there exists an ergodic measure* $\mu_\alpha \in \mathcal{M}_\Phi(\Lambda)$ *such that* $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, *and* $\dim_u \mu_\alpha = \mathcal{F}_u(\alpha)$.

Moreover, the spectrum $\mathcal{F}_u$ *is analytic in* $\operatorname{int} \mathcal{P}(\mathcal{M}_\Phi(\Lambda))$.

4 Dimension Spectra

4.1 Main Result

Let $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ be a C^1 flow in a smooth manifold M and let $\Lambda \subset M$ be a hyperbolic set. The flow Φ is said to be *conformal* on Λ if the maps

$$d_x\varphi_t|E^s(x)\colon E^s(x) \to E^s(\varphi_t(x)) \quad \text{and} \quad d_x\varphi_t|E^u(x)\colon E^u(x) \to E^u(\varphi_t(x))$$

are multiples of isometries for every $x \in \Lambda$ and $t \in \mathbb{R}$. We shall consider the functions

$$v(x) = \frac{\partial}{\partial t} \log\|d_x\varphi_t|E^u(x)\|\Big|_{t=0}$$

and

$$w(x) = -\frac{\partial}{\partial t} \log\|d_x\varphi_t|E^s(x)\|\Big|_{t=0}.$$

Given pairs of functions $(A^\pm, B^\pm) \in F(\Lambda)$ (with the symbols $+$ and $-$ corresponding respectively to the future and to the past), we write

$$A^+ = (a_1^+, \dots, a_d^+), \qquad B^+ = (b_1^+, \dots, b_d^+),$$

and

$$A^- = (a_1^-, \dots, a_d^-), \qquad B^- = (b_1^-, \dots, b_d^-).$$

We assume that $B^\pm > 0$. Given $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ in $\mathbb{R}^d$, we set

$$K_\alpha^+ = \bigcap_{i=1}^d \left\{ x \in \Lambda : \lim_{t\to+\infty} \frac{\int_0^t a_i^+(\varphi_s(x))\,ds}{\int_0^t b_i^+(\varphi_s(x))\,ds} = \alpha_i \right\},$$

and

$$K_\beta^- = \bigcap_{i=1}^d \left\{ x \in \Lambda : \lim_{t\to-\infty} \frac{\int_0^t a_i^-(\varphi_s(x))\,ds}{\int_0^t b_i^-(\varphi_s(x))\,ds} = \beta_i \right\}.$$

The *dimension spectrum* $\mathcal{D}\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\mathcal{D}(\alpha, \beta) = \dim_H(K_\alpha^+ \cap K_\beta^-).$$

The following is our main result.

Theorem 1 *Let* Λ *be a locally maximal hyperbolic set of a* $C^{1+\alpha}$ *flow* Φ *such that* Φ *is topologically mixing and conformal on* Λ, *and let* $(A^\pm, B^\pm) \in F(\Lambda)$. *Then the following properties hold*:

1. *if*

$$\alpha \in \operatorname{int} \mathcal{P}^+(\mathcal{M}_\Phi(\Lambda)) \quad \textit{and} \quad \beta \in \operatorname{int} \mathcal{P}^-(\mathcal{M}_\Phi(\Lambda)), \tag{7}$$

then

$$\begin{aligned}\mathcal{D}(\alpha, \beta) &= \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda \\ &= \max\left\{\frac{h_\mu(\Phi)}{\int_\Lambda v\,d\mu} : \mu \in \mathcal{M}_\Phi(\Lambda) \textit{ and } \mathcal{P}^+(\mu) = \alpha\right\} \\ &\quad + \max\left\{\frac{h_\mu(f)}{\int_\Lambda w\,d\mu} : \mu \in \mathcal{M}_\Phi(\Lambda) \textit{ and } \mathcal{P}^-(\mu) = \beta\right\} + 1;\end{aligned}$$

2. *the spectrum* $\mathcal{D}$ *is analytic in* $\operatorname{int} \mathcal{P}^+(\mathcal{M}_\Phi(\Lambda)) \times \operatorname{int} \mathcal{P}^-(\mathcal{M}_\Phi(\Lambda))$.

We separate the proof of Theorem 1 into several steps. In the remaining sections the assumptions in the theorem are standing assumptions.

4.2 Preliminary Results

For each $x \in \Lambda$ there exist *local stable* and *unstable manifolds* $V^s(x)$ and $V^u(x)$ containing x such that:

1. $T_x V^s(x) = E^s(x)$ and $T_x V^u(x) = E^u(x)$;
2. for every $t > 0$ we have

$$\varphi_t(V^s(x)) \subset V^s(\varphi_t(x)) \quad \text{and} \quad \varphi_{-t}(V^u(x)) \subset V^u(\varphi_{-t}(x));$$

3. there exist $\kappa > 0$ and $\mu \in (0, 1)$ such that for every $t > 0$ we have

$$d(\varphi_t(y), \varphi_t(x)) \le \kappa \mu^t d(y, x) \quad \text{if } y \in V^s(x), \tag{8}$$

and

$$d(\varphi_{-t}(y), \varphi_{-t}(x)) \le \kappa \mu^t d(y, x) \quad \text{if } y \in V^u(x).$$

The following is a preliminary result along stable and unstable manifolds.

Lemma 1 *For each* $\alpha, \beta \in \mathbb{R}^d$, $x^+ \in K_\alpha^+$, *and* $x^- \in K_\beta^-$ *we have*

$$\dim_H K_\alpha^+ = \dim_H (K_\alpha^+ \cap V^u(x^+)) + t_s + 1 = \dim_v K_\alpha^+ + t_s + 1,$$

and

$$\dim_H K_\beta^- = \dim_H (K_\beta^- \cap V^s(x^-)) + t_u + 1 = \dim_w K_\beta^- + t_u + 1,$$

where t_s *and* t_u *are the unique real numbers such that*

$$P_{\Phi|\Lambda}(-t_s v) = P_{\Phi|\Lambda}(-t_u w) = 0.$$

Proof It follows easily from (8) and the uniform continuity of $a_i^\pm$ and $b_i^\pm$ in Λ that $V^s(x) \subset K_\alpha^+$ for every $x \in K_\alpha^+$, and thus also

$$\bigcup_{t \in \mathbb{R}} \varphi_t(V^s(x)) \subset K_\alpha^+$$

for every $x \in K_\alpha^+$, since the set K_α^+ is Φ-invariant.

Since Φ is conformal on Λ, it follows from results of Hasselblatt in [20] that the distributions $x \mapsto E^s(x) \oplus E^0(x)$ and $x \mapsto E^u(x)$ are Lipschitz. Therefore, for a sufficiently small open neighborhood of a point $x \in K_\alpha^+$ there is a Lipschitz map with Lipschitz inverse from the set K_α^+ to

$$\bigcup_{t\in I} \varphi_t(V^s(x)) \times V^u(x),$$

where I is an open interval containing zero. This implies that

$$\dim_H K_\alpha^+ = \dim_H(K_\alpha^+ \cap V^u(x)) + t_s + 1,$$

since the Hausdorff dimension and the upper box dimension of $K_\alpha^+ \cap V^u(x)$ coincide (due to the conformality of Φ on Λ).

For the second identity, we note that

$$\int_0^t v(\varphi_s x)\, ds = \log \|d_x\varphi_t | E^u(x)\|.$$

Since the distribution $x \mapsto E^u(x)$ is Lipschitz and Φ is of class $C^{1+\alpha}$, the function v is Hölder continuous and for each $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that

$$c_1 \exp(-\alpha v(x,t,\varepsilon)) \le [\operatorname{diam}(B_\varepsilon(x,t) \cap V^u(x))]^\alpha \le c_2 \exp(-\alpha v(x,t,\varepsilon)).$$

By the definition of Hausdorff dimension, this readily implies that for every set $Z \subset \Lambda$ we have

$$\dim_H(Z \cap V^u(x)) = \dim_v Z.$$

To obtain the second identity in the lemma we take $Z = K_\alpha^+$.

The arguments for K_β^- are entirely analogous. □

4.3 Markov Systems and Reduction to Discrete Time

We need some additional material for the remaining arguments of the proof.

Since Λ is locally maximal, for each sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ are at a distance $d(x,y) \le \delta$, then there exists a unique $t = t(x,y) \in [-\varepsilon, \varepsilon]$ such that the set $[x,y] := V^s(\varphi_t(x)) \cap V^u(y)$ consists of a single point in Λ.

Let $D \subset M$ be an open disk of dimension $\dim M - 1$ which is transverse to the flow Φ. Given $x \in D$, there exists a diffeomorphism from $D \times (-\varepsilon, \varepsilon)$ onto an open neighborhood $U(x)$ of x. The projection map $\pi_D \colon U(x) \to D$ defined by $\pi_D(\varphi_t(y)) = y$ is differentiable.

A closed set $R \subset \Lambda \cap D$ is called a *rectangle* if $R = \overline{\operatorname{int} R}$ (where the interior is computed with respect to the induced topology in $\Lambda \cap D$), and $\pi_D([x,y]) \in R$ whenever $x, y \in R$. Consider a collection of rectangles $R_1, \dots, R_p \subset \Lambda$ (each contained in some disk transverse to the flow) with $R_i \cap R_j = \partial R_i \cap \partial R_j$ whenever $i \ne j$, and assume that there exists $\varepsilon > 0$ such that:

1. $\Lambda = \bigcup_{t\in[0,\varepsilon]} \varphi_t(P)$, where $P = \bigcup_{i=1}^p R_i$;
2. for each $i \ne j$ either $\varphi_t(R_i) \cap R_j = \emptyset$ for all $t \in [0,\varepsilon]$ or $\varphi_t(R_j) \cap R_i = \emptyset$ for all $t \in [0,\varepsilon]$.

We define the *transfer function* $\tau \colon \Lambda \to [0, \infty)$ by

$$\tau(x) = \min\left\{t > 0 : \varphi_t(x) \in P\right\}. \tag{9}$$

Then the *transfer map* $T \colon \Lambda \to P$ is defined by $T(x) = \varphi_{\tau(x)}(x)$. We note that the restriction of T to P is invertible.

We say that the rectangles $R_1, \dots, R_p$ form a *Markov system* for Φ on the set Λ if

$$T(\operatorname{int}(V^s(x) \cap R_i)) \subset \operatorname{int}(V^s(T(x)) \cap R_j),$$

and

$$T^{-1}(\operatorname{int}(V^u(T(x)) \cap R_j)) \subset \operatorname{int}(V^u(x) \cap R_i)$$

whenever $x \in \operatorname{int} T(R_i) \cap \operatorname{int} R_j$. Any locally maximal hyperbolic set Λ has Markov systems of arbitrarily small diameter (see [10, 40]). Furthermore, the map τ is Hölder continuous on each domain of continuity, and

$$0 < \inf_{x \in \Lambda} \tau \le \sup_{x \in \Lambda} \tau < \infty.$$

Given a Markov system $R_1, \dots, R_p$ for Φ on Λ we define a $p \times p$ matrix $A = (a_{ij})$ with $a_{ij} = 1$ if $\operatorname{int} T(R_i) \cap \operatorname{int} R_j \ne \emptyset$, and $a_{ij} = 0$ otherwise. We consider the set $\Sigma_A \subset \{1, \dots, p\}^{\mathbb{Z}}$ given by

$$\Sigma_A = \left\{(\cdots i_{-1} i_0 i_1 \cdots) : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{Z}\right\},$$

and the shift map $\sigma \colon \Sigma_A \to \Sigma_A$ defined by $\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for every $n \in \mathbb{Z}$. Given $\beta > 1$, we equip Σ_A with the distance d_β defined by

$$d_\beta\big((\cdots i_{-1} i_0 i_1 \cdots), (\cdots j_{-1} j_0 j_1 \cdots)\big) = \sum_{n=-\infty}^{\infty} \beta^{-|n|} |i_n - j_n|.$$

We define a *coding map* $\pi \colon \Sigma_A \to P$ for the set $\Lambda \cap P$ by

$$\pi(\cdots i_0 \cdots) = \bigcap_{j \in \mathbb{Z}} \overline{T^{-j}(\operatorname{int} R_{i_j})}.$$

One can easily verify that $\pi \circ \sigma = T \circ \pi$. As observed in [10], it is always possible to choose β so that the function $\tau \circ \pi \colon \Sigma_A \to [0, \infty)$ is Lipschitz.

We denote by Σ_A^+ the set of sequences $(i_0 i_1 \cdots)$ such that

$$(i_0 i_1 \cdots) = (j_0 j_1 \cdots) \quad \text{for some } (\cdots j_{-1} j_0 j_1 \cdots) \in \Sigma_A,$$

and by Σ_A^- the set of sequences $(\cdots i_{-1} i_0)$ such that

$$(\cdots i_{-1} i_0) = (\cdots j_{-1} j_0) \quad \text{for some } (\cdots j_{-1} j_0 j_1 \cdots) \in \Sigma_A.$$

We note that Σ_A^- is identified with $\Sigma_{A^t}^+$, where A^t denotes the transpose of A, by the map

$$\Sigma_A^- \ni (\cdots i_{-1} i_0) \mapsto (i_0 i_{-1} \cdots) \in \Sigma_{A^t}^+.$$

We also consider the shift maps $\sigma^+ \colon \Sigma_A^+ \to \Sigma_A^+$ and $\sigma^- \colon \Sigma_A^- \to \Sigma_A^-$ defined by

$$\sigma^+(i_0 i_1 \cdots) = (i_1 i_2 \cdots) \quad \text{and} \quad \sigma^-(\cdots i_{-1} i_0) = (\cdots i_{-2} i_{-1}).$$

Now let $\pi^+ \colon \Sigma_A \to \Sigma_A^+$ and $\pi^- \colon \Sigma_A \to \Sigma_A^-$ be the projections defined by

$$\pi^+(\cdots i_{-1} i_0 i_1 \cdots) = (i_0 i_1 \cdots) \quad \text{and} \quad \pi^-(\cdots i_{-1} i_0 i_1 \cdots) = (\cdots i_{-1} i_0).$$

Given $x \in P$ we choose $\omega \in \Sigma_A$ such that $\pi(\omega) = x$. Let $R(x)$ be a rectangle of the Markov system which contains x. For each $\omega' \in \Sigma_A$ we have

$$\pi(\omega') \in V^u(x) \cap R(x) \quad \text{whenever } \pi^-(\omega') = \pi^-(\omega),$$

and

$$\pi(\omega') \in V^s(x) \cap R(x) \quad \text{whenever } \pi^+(\omega') = \pi^+(\omega).$$

Therefore, writing $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$, the set $V^u(x) \cap R(x)$ can be identified with the cylinder set

$$C_{i_0}^+ = \{(j_0 j_1 \cdots) \in \Sigma_A^+ : j_0 = i_0\} \subset \Sigma_A^+, \tag{10}$$

and the set $V^s(x) \cap R(x)$ can be identified with the cylinder set

$$C_{i_0}^- = \{(\cdots j_{-1} j_0) \in \Sigma_A^- : j_0 = i_0\} \subset \Sigma_A^-. \tag{11}$$

Given a continuous function $a \colon \Lambda \to \mathbb{R}$ and a Markov system for the flow Φ on Λ, we define the function $I_a \colon \Lambda \to \mathbb{R}$ by

$$I_a(x) = \int_0^{\tau(x)} a(\varphi_s(x))\, ds,$$

with τ as in (9). We recall that a function $A \colon \Lambda \to \mathbb{R}$ is said to be *T-cohomologous* to a function $B \colon \Lambda \to \mathbb{R}$ if there is a bounded measurable function $q \colon \Lambda \to \mathbb{R}$ such that

$$A - B = q \circ T - q.$$

Lemma 2 [7] *Let $a, b \colon \Lambda \to \mathbb{R}$ be continuous functions. Then the following properties are equivalent*:

1. *a is Φ-cohomologous to b and* (6) *holds for every $x \in \Lambda$*;
2. *I_a is T-cohomologous to I_b and*

$$I_a(x) - I_b(x) = q(T(x)) - q(x) \textit{ for every } x \in \Lambda.$$

It is also of interest to characterize the convergence of the Birkhoff averages of the flow Φ in terms of the transfer map T.

Lemma 3 [7] *Given a continuous function $a \colon \Lambda \to \mathbb{R}$, the following properties hold*:

1. *if $a \colon \Lambda \to \mathbb{R}$ is Hölder continuous, then I_a is Hölder continuous on each domain of continuity of τ*;

2. *if $x \in \Lambda$, then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t a(\varphi_s(x))\,ds = \liminf_{m\to\infty} \frac{\sum_{i=0}^m I_a(T^i(x))}{\sum_{i=0}^m \tau(T^i(x))},$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t a(\varphi_s(x))\,ds = \limsup_{m\to\infty} \frac{\sum_{i=0}^m I_a(T^i(x))}{\sum_{i=0}^m \tau(T^i(x))}.$$

4.4 Construction of Auxiliary Measures

The following statement is a consequence of a construction described by Bowen in [11, Lemma 1.6].

Lemma 4 *For each $i, j = 1, \dots, d$ there exist Hölder continuous functions*

$$a_i^u, b_i^u, d^u \colon \Sigma_A^+ \to \mathbb{R} \quad \textit{and} \quad a_j^s, b_j^s, d^s \colon \Sigma_A^- \to \mathbb{R},$$

and continuous functions $g_i^+, h_i^+, g_j^-, h_j^-, \rho^\pm \colon \Sigma_A \to \mathbb{R}$ such that

$$\begin{aligned} I_{a_i^+} \circ \pi &= a_i^u \circ \pi^+ + g_i^+ - g_i^+ \circ \sigma,\\ I_{b_i^+} \circ \pi &= b_i^u \circ \pi^+ + h_i^+ - h_i^+ \circ \sigma,\\ I_v \circ \pi &= d^u \circ \pi^+ + \rho^+ - \rho^+ \circ \sigma, \end{aligned}$$

and

$$\begin{aligned} I_{a_j^-} \circ \pi &= a_j^s \circ \pi^- + g_j^- - g_j^- \circ \sigma,\\ I_{b_j^-} \circ \pi &= b_j^s \circ \pi^- + h_j^- - h_j^- \circ \sigma,\\ I_w \circ \pi &= d^s \circ \pi^- + \rho^- - \rho^- \circ \sigma. \end{aligned}$$

We write

$$A^u = \left(a_1^u, \dots, a_d^u\right), \qquad B^u = \left(b_1^u, \dots, b_d^u\right),$$

and

$$A^s = \left(a_1^s, \dots, a_d^s\right), \qquad B^s = \left(b_1^s, \dots, b_d^s\right).$$

Given $q^\pm \in \mathbb{R}^d$, we define Hölder continuous functions $U \colon \Sigma_A^+ \to \mathbb{R}$ and $S \colon \Sigma_A^- \to \mathbb{R}$ by

$$\begin{aligned} U &= \langle q^+, A^u - \alpha * B^u \rangle - d^+ d^u,\\ S &= \langle q^-, A^s - \beta * B^s \rangle - d^- d^s, \end{aligned} \tag{12}$$

where

$$d^+ = \dim_H K_\alpha^+ - t_s - 1 \quad \text{and} \quad d^- = \dim_H K_\beta^- - t_u - 1. \tag{13}$$

Now let μ^u be the equilibrium measure of U in Σ_A^+ (with respect to σ^+), and let μ^s be the equilibrium measure of S in Σ_A^- (with respect to σ^-). The following is an immediate consequence of statements 2 and 3 in Proposition 5.

Lemma 5 *If* (7) *holds, then there exist* $q^{\pm} \in \mathbb{R}^d$ *such that*

$$P_{\sigma^+}(U) = P_{\sigma^-}(S) = 0,$$
$$\int_{\Sigma_A^+} A^u \, d\mu^u = \alpha * \int_{\Sigma_A^+} B^u \, d\mu^u,$$

and

$$\int_{\Sigma_A^-} A^u \, d\mu^s = \beta * \int_{\Sigma_A^-} B^s \, d\mu^s.$$

Given $x \in P$, let again $R(x)$ be a rectangle of the Markov system which contains x. We consider the measures in $R(x)$ defined by

$$\nu^u = \mu^u \circ \pi^+ \circ \pi^{-1} \quad \text{and} \quad \nu^s = \mu^s \circ \pi^- \circ \pi^{-1},$$

using the vectors $q^{\pm}$ in Lemma 5. Finally, we define a measure ν in $R(x)$ by $\nu = \nu^u \times \nu^s$. Since μ^u and μ^s are Gibbs measures we have

$$\nu(R(x)) = \mu^u\big(C_{i_0}^+\big)\mu^s\big(C_{i_0}^-\big) > 0,$$

with $C_{i_0}^+$ and $C_{i_0}^-$ as in (10) and (11). In the following sections we establish several properties of the measure ν.

4.5 Lower Pointwise Dimension

Lemma 6 *For* ν*-almost every* $x \in P$ *we have*

$$\liminf_{r \to 0} \frac{\log \nu(B(x,r) \cap P)}{\log r} \geq \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1.$$

Proof We follow arguments in the proof of Lemma 4 in [9].

By the variational principle for the topological pressure applied to the functions U and S in (12), and Lemma 5, we obtain

$$\frac{h_{\mu^u}(\sigma^+)}{\int_{\Sigma_A^+} d^u \, d\mu^u} = d^+ \quad \text{and} \quad \frac{h_{\mu^s}(\sigma^-)}{\int_{\Sigma_A^-} d^s \, d\mu^s} = d^-.$$

By Shannon–McMillan–Breiman's theorem and Birkhoff's ergodic theorem, for each $\varepsilon > 0$, μ^s-almost every $\omega^+ \in C_{i_0}^+$, and μ^u-almost every $\omega^- \in C_{i_0}^-$ there exists $s(\omega) \in \mathbb{N}$ (with $\omega^+ = \pi^+(\omega)$ and $\omega^- = \pi^-(\omega)$) such that for every $n, m > s(\omega)$ we have

$$d^+ - \varepsilon < -\frac{\log \mu^u(C_{i_0 \cdots i_n}^+)}{\sum_{k=0}^n d^u((\sigma^+)^k(\omega^+))} < d^+ + \varepsilon,$$

and

$$d^- - \varepsilon < -\frac{\log \mu^s(C_{i_{-m} \cdots i_0}^-)}{\sum_{k=0}^m d^s((\sigma^-)^k(\omega^-))} < d^- + \varepsilon.$$

For each sufficiently small $r > 0$, let $n = n(\omega, r)$ and $m = m(\omega, r)$ be the unique positive integers such that

$$-\sum_{k=0}^{n} d^u((\sigma^+)^k(\omega^+)) > \log r, \qquad -\sum_{k=0}^{n+1} d^u((\sigma^+)^k(\omega^+)) \le \log r, \tag{14}$$

and

$$-\sum_{k=0}^{m} d^s((\sigma^-)^k(\omega^-)) > \log r, \qquad -\sum_{k=0}^{m+1} d^s((\sigma^-)^k(\omega^-)) \le \log r. \tag{15}$$

By Lemma 6.1 in [36] there exists $\rho > 1$ (independent of $x = \pi(\omega)$ and r) such that

$$B(y, r/\rho) \cap P \subset \pi(C_{i_{-m}\cdots i_n}) \subset B(x, \rho r) \cap P \tag{16}$$

for some point $y \in \pi(C_{i_{-m}\cdots i_n})$, where $\omega = (\cdots i_{-1} i_0 i_1 \cdots)$. Moreover, by Lemma 1 in [7], for ν-almost every $y \in P$ there exist $\eta = 2\rho$ and $\delta = \delta(\eta, y, \varepsilon) > 0$ such that

$$\nu(B(y, \eta r) \cap P) \le \nu(B(y, r) \cap P) r^{-\varepsilon}$$

for every $r < \delta$ (since all probability measures in $\mathbb{R}^n$ are weakly diametrically regular). We obtain

$$\begin{aligned}
\nu(B(x, r) \cap P) &\le \nu\left(B\left(y, 2\rho \frac{r}{\rho}\right) \cap P\right) \\
&\le \nu(B(y, r/\rho) \cap P) \left(\frac{r}{\rho}\right)^{-\varepsilon} \\
&\le \nu(\pi(C_{i_{-m}\cdots i_n})) \left(\frac{r}{\rho}\right)^{-\varepsilon} \\
&= \mu^u(C^+_{i_0\cdots i_n}) \mu^s(C^-_{i_{-m}\cdots i_0}) \left(\frac{r}{\rho}\right)^{-\varepsilon} \\
&\le \exp\left[(-d^+ + \varepsilon) \sum_{k=0}^{n} d^u((\sigma^+)^k(\omega^+))\right] \\
&\quad \times \exp\left[(-d^- + \varepsilon) \sum_{k=0}^{m} d^s((\sigma^-)^k(\omega^-))\right] \left(\frac{r}{\rho}\right)^{-\varepsilon} \\
&\le \exp[(\log r + \|d^u\|_\infty)(d^+ - \varepsilon)] \\
&\quad \times \exp[(\log r + \|d^s\|_\infty)(d^- - \varepsilon)] \left(\frac{r}{\rho}\right)^{-\varepsilon},
\end{aligned}$$

and hence,

$$\liminf_{r \to 0} \frac{\log \nu(B(x, r) \cap P)}{\log r} \ge d^+ + d^- - 2\varepsilon \tag{17}$$

for ν-almost every $x \in P$.

On the other hand, by Theorem 4.2 in [36] we have

$$\dim_H \Lambda = t_s + t_u + 1. \tag{18}$$

Therefore, by (13) and (17) we obtain

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r)\cap P)}{\log r} \geq \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1 - 2\varepsilon,$$

and the result follows from the arbitrariness of ε. □

4.6 Upper Pointwise Dimension

Lemma 7 *For every $x \in K_\alpha^+ \cap K_\beta^- \cap P$ we have*

$$\limsup_{r\to 0} \frac{\log \nu(B(x,r)\cap P)}{\log r} \leq \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1.$$

Proof We follow arguments in the proofs of Lemmas 5 and 6 in [9]. Let $x \in K_\alpha^+ \cap K_\beta^- \cap P$. Let also $\omega \in \Sigma_A$ be such that $\pi(\omega) = x$ and consider the projections $\omega^\pm = \pi^\pm(\omega)$. It follows from Lemma 4 that

$$\begin{aligned} I_{a_i^+}(T^k(\pi(\omega))) &= I_{a_i^+}(\pi(\sigma^+(\omega))) \\ &= a_i^u(\pi^+(\sigma^k(\omega))) + g_i^+(\sigma^k(\omega)) - g_i^+(\sigma^{k+1}(\omega)) \\ &= a_i^u((\sigma^+)^k(\omega^+)) + g_i^+(\sigma^k(\omega)) - g_i^+(\sigma^{k+1}(\omega)), \end{aligned}$$

with analogous identities for the functions $I_{b_i^+}$, $I_{a_j^-}$ and $I_{b_j^-}$. Therefore,

$$\frac{\sum_{k=0}^{n-1} I_{a_i^+}(T^k(x))}{\sum_{k=0}^{n-1} I_{b_i^+}(T^k(x))} = \frac{\sum_{k=0}^{n-1} a_i^u((\sigma^+)^k(\omega^+)) + g_i^+(\omega) - g_i^+(\sigma^n(\omega))}{\sum_{k=0}^{n-1} b_i^u((\sigma^+)^k(\omega^+)) + h_i^+(\omega) - h_i^+(\sigma^n(\omega))},$$

and

$$\frac{\sum_{k=0}^{n-1} I_{a_j^-}(T^k(x))}{\sum_{k=0}^{n-1} I_{b_j^-}(T^k(x))} = \frac{\sum_{k=0}^{n-1} a_j^s((\sigma^-)^k(\omega^-)) + g_j^-(\omega) - g_j^-(\sigma^n(\omega))}{\sum_{k=0}^{n-1} b_j^s((\sigma^-)^k(\omega^-)) + h_j^-(\omega) - h_j^-(\sigma^n(\omega))}.$$

On the other hand,

$$\sum_{k=0}^{n-1} b_i^u((\sigma^+)^k(\omega^+)) \geq n \inf b_i^+ \inf \tau - 2\|h_i^+\|_\infty,$$

and

$$\sum_{k=0}^{n-1} b_j^s((\sigma^-)^k(\omega^-)) \geq n \inf b_j^- \inf \tau - 2\|h_j^-\|_\infty.$$

Since $b_i^+, b_j^- > 0$ and $\inf \tau > 0$ this ensures that the limits

$$\frac{\sum_{k=0}^{n-1} I_{a_i^+}(T^k(x))}{\sum_{k=0}^{n-1} I_{b_i^+}(T^k(x))}, \qquad \frac{\sum_{k=0}^{n-1} I_{a_j^-}(T^k(x))}{\sum_{k=0}^{n-1} I_{b_j^-}(T^k(x))}$$

exist if and only if the limits

$$\frac{\sum_{k=0}^{n-1} a_i^u((\sigma^+)^k(\omega^+))}{\sum_{k=0}^{n-1} b_i^u((\sigma^+)^k(\omega^+))}, \qquad \frac{\sum_{k=0}^{n-1} a_j^s((\sigma^-)^k(\omega^-))}{\sum_{k=0}^{n-1} b_j^s((\sigma^-)^k(\omega^-))},$$

exist, in which case they are respectively equal.

By Lemma 3, if $x \in K_\alpha^+ \cap K_\beta^- \cap P$ and $\omega \in \Sigma_A$ are such that $\pi(\omega) = x$, then given $\varepsilon > 0$ there exists $r(\omega) \in \mathbb{N}$ such that for every $n > r(\omega)$ we have

$$\left\| \left\langle q^+, \sum_{k=0}^{n} (A^u - \alpha * B^u)((\sigma^+)^k(\omega^+)) \right\rangle \right\| < \varepsilon n \|\langle q^+, B^u\rangle\|_\infty,$$

and

$$\left\| \left\langle q^-, \sum_{k=0}^{n} (A^s - \beta * B^s)((\sigma^-)^k(\omega^-)) \right\rangle \right\| < \varepsilon n \|\langle q^-, B^s\rangle\|_\infty.$$

By Lemma 5, we have $P_{\sigma^+}(U) = 0$ and since μ^u is a Gibbs measure, there exists $D > 0$ such that for every $i_0 = 1, \dots, p$ and $n \in \mathbb{N}$ we have

$$D^{-1} < \frac{\mu^u(C^+_{i_0\cdots i_n})}{\exp \sum_{k=0}^{n} U((\sigma^+)^k(\omega^+))} < D,$$

and thus,

$$\mu^u(C^+_{i_0\cdots i_n}) > D^{-1} \exp\left[-d^+ \sum_{k=0}^{n} d^u((\sigma^+)^k(\omega^+)) - \varepsilon n \|\langle q^+, B^u\rangle\|_\infty \right]. \tag{19}$$

Similarly, for every $i_0 = 1, \dots, p$ and $n \in \mathbb{N}$ we have

$$\mu^s(C^-_{i_{-m}\cdots i_0}) > D^{-1} \exp\left[-d^- \sum_{k=0}^{m} d^s((\sigma^-)^k(\omega^-)) - \varepsilon m \|\langle q^-, B^s\rangle\|_\infty \right], \tag{20}$$

eventually taking a larger D. By the hyperbolicity of Φ on Λ, and since $\inf_{x\in\Lambda} \tau > 0$, there exists $r > 0$ such that $n(\omega, r) > r(\omega)$ and $m(\omega, r) > r(\omega)$ (see (14) and (15)). Moreover, by (16) there exists $\rho > 0$ (independent of $x = \pi(\omega)$ and r) such that

$$B(x, \rho r) \cap P \supset \pi(C_{i_{-m}\cdots i_n}),$$

where $n = n(\omega, r)$ and $m = m(\omega, r)$. Combining (19) and (20) with (14) and (15) we obtain

$$\begin{aligned}\nu(B(x, \rho r) \cap P) &\ge \nu(\pi(C_{i_{-m}\cdots i_n})) = \mu^u(C^+_{i_0\cdots i_n})\mu^s(C^-_{i_{-m}\cdots i_0}) \\ &\ge D^{-2} r^{d^+ + d^-} \exp(-\varepsilon n \|\langle q^+, B^u\rangle\|_\infty - \varepsilon m \|\langle q^-, B^s\rangle\|_\infty),\end{aligned}$$

for every sufficiently small $r > 0$. On the other hand, it follows from (14) and (15) that

$$-n \inf d^u > \log r, \qquad -m \inf d^s > \log r.$$

Therefore, for every $x \in K_\alpha^+ \cap K_\beta^- \cap P$ we have

$$\limsup_{r\to\infty} \frac{\log \nu(B(x, r) \cap P)}{\log r} \le d^+ + d^- + \varepsilon \left(\frac{\|\langle q^+, B^u\rangle\|_\infty}{\inf d^u} + \frac{\|\langle q^-, B^s\rangle\|_\infty}{\inf d^s} \right).$$

Since ε can be made arbitrarily small, we obtain

$$\limsup_{r\to\infty} \frac{\log \nu(B(x,r)\cap P)}{\log r} \le d^+ + d^-.$$

Together with (13) and (18) this yields the desired result. □

4.7 Proof of Theorem 1

As a consequence of the above lemmas we have the following result.

Lemma 8 *If* (7) *holds, then there exists a probability measure* ν *in* P *such that* $\nu(K_\alpha^+ \cap K_\beta^- \cap P) = 1$,

$$\lim_{r\to\infty} \frac{\log \nu(B(x,r)\cap P)}{\log r} = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1 \tag{21}$$

for ν*-almost every* $x \in P$, *and*

$$\limsup_{r\to\infty} \frac{\log \nu(B(x,r)\cap P)}{\log r} \le \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1 \tag{22}$$

for every $x \in K_\alpha^+ \cap K_\beta^- \cap P$.

We can now establish our main result.

Proof of Theorem 1 Let ν be the measure in Lemma 8. It follows from (21) (see for example [1, Theorem 2.1.5]) that

$$\dim_H \nu = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1,$$

where

$$\dim_H \nu = \inf\big\{\dim_H Z : \nu(Z) = 1\big\}.$$

Since $\nu(K_\alpha^+ \cap K_\beta^- \cap P) = 1$ we obtain

$$\dim_H (K_\alpha^+ \cap K_\beta^- \cap P) \ge \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1.$$

On the other hand, it follows from (22) (see for example [1, Theorem 2.1.5]) that

$$\dim_H (K_\alpha^+ \cap K_\beta^- \cap P) \le \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1,$$

and thus,

$$\dim_H (K_\alpha^+ \cap K_\beta^- \cap P) = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda - 1.$$

Since $K_\alpha^+ \cap K_\beta^-$ is locally diffeomorphic to a product between $K_\alpha^+ \cap K_\beta^- \cap P$ and an interval, we obtain

$$\mathcal{D}(\alpha, \beta) = \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda.$$

By Lemma 1, (18), and statement 1 in Proposition 5 we conclude that

$$\begin{aligned}\mathcal{D}(\alpha,\beta) &= \dim_H(K_\alpha^+ \cap V^u(x)) + \dim_H(K_\beta^- \cap V^s(x)) + 1\\ &= \dim_u K_\alpha^+ + \dim_w K_\beta^- + 1\\ &= \max\left\{\frac{h_\mu(\Phi)}{\int_\Lambda v\,d\mu} : \mu \in \mathcal{M}_\Phi(\Lambda) \text{ and } \mathcal{P}^+(\mu) = \alpha\right\}\\ &\quad + \max\left\{\frac{h_\mu(\Phi)}{\int_\Lambda w\,d\mu} : \mu \in \mathcal{M}_\Phi(\Lambda) \text{ and } \mathcal{P}^-(\mu) = \beta\right\} + 1.\end{aligned}$$

The second statement is now an immediate consequence of the last statement in Proposition 5. □

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